Economics 808  
Advanced macroeconomic theory  
Fall 2014  

Lecture 9: Ramsey optimal taxation in the growth model  

1 The deterministic case  

1.1 Preliminaries  

Apart from this text, you may want to read Chari and Kehoe (1998).  

The problem of optimal taxation—how to raise a given a amount of revenue at least social cost—can be approached in different ways. The approach here, following Ramsey (1927) is to suppose that taxes must be proportional to income and/or consumption. In particular, lump-sum taxes are ruled out. No theoretical foundations are given for this particular taxation scheme.  

More precisely, the solution concept is the following. The Ramsey optimal allocation is that allocation that delivers the highest weighted sum of utilities among those allocations that form part of some competitive equilibrium. Notice that the proportionality of taxes is built in the concept of competitive equilibrium. Non-proportional taxes mean that people face different after-tax prices, and that is not consistent with competitive equilibrium.  

A competitive equilibrium consists of an allocation and after-tax prices. Alternatively, and perhaps more intuitively, we can think of it as having three parts: an allocation, pre-tax prices
and tax rates. In this context it is important to understand that a given allocation is not associated with a unique set of tax rates. It can easily happen that the same equilibrium allocation is supported by distinct tax rates. This phenomenon is known as *tax equivalence*. To see how this might work, consider an environment where a representative agent maximizes

$$\sum_{t=0}^{\infty} \beta^t u(c_t, \ell_t)$$

subject to

$$h_t = 1 - \ell_t$$

$$(1 + \tau^c_t) c_t + k_{t+1} = (1 - \tau^k_t) r_t k_t + (1 - \tau^h_t) w_t h_t,$$

$k_0$ given and some suitable No Ponzi Scheme constraint.

The agent’s first order conditions are

$$(1 - \tau^h_t) w_t u_{c,t} = (1 + \tau^c_t) u_{\ell,t}$$

and

$$(1 + \tau^c_{t+1}) u_{c,t} = (1 + \tau^c_t) u_{c,t+1} (1 - \tau^k_{t+1}) r_{t+1}.$$  

Now imagine an equilibrium where $\tau^c_t = 0$, $\tau^h_t = \tau^h$ and $\tau^k_t = \tau^k$. Suppose now that capital taxes are deemed unconstitutional so that $\tilde{\tau}^k_t = 0$. Can we now redesign the other taxes so that the previous equilibrium allocation is still an optimal choice?

Here’s how to do it. In the first equation, what matters is the ratio $(1 - \tau^h_t)/(1 + \tau^c_t)$. Let’s make sure that this is unchanged, so that

$$\frac{1 - \tilde{\tau}^h_t}{1 + \tilde{\tau}^c_t} = 1 - \tau^h$$

and, similarly,

$$\frac{1 + \tilde{\tau}^c_{t+1}}{1 + \tilde{\tau}^c_t} = \frac{1}{1 - \tau^k}.$$

Clearly this leads to no contradiction. Indeed, we even have an extra degree of freedom. Set $\tilde{\tau}^c_t$ arbitrarily; for instance, set it to zero. To satisfy the second equation, we have to set

$$1 + \tilde{\tau}^c_{t+1} = \frac{1 + \tilde{\tau}^c_t}{1 - \tau^k}$$

or, when $\tilde{\tau}^c_0$,

$$1 + \tilde{\tau}^c_t = (1 - \tau^k)^{-t}.$$
If $\tau^k > 0$ this implies an ever increasing consumption tax. Meanwhile,

$$1 - \tilde{\tau}_t^h = (1 + \tilde{\tau}_t^c)(1 - \tau^h).$$

You may wonder if this tilde tax scheme is truly equivalent to the original one in the sense that it balances the government’s intertemporal budget constraint. Well, with an arbitrary choice of $\tau_0^c$ it doesn’t necessarily. But if we are free to manipulate that too—using up that one degree of freedom—we can use it to balance the government’s intertemporal budget.

### 1.2 The primal approach

I hope that the previous section convinced you that it is better to try to eliminate taxes and go for the Ramsey optimal allocation directly, without taking a stand on precisely how that allocation is enforced as a competitive equilibrium. The question is how to do that.

In a way, the dual approach is more straightforward. Let allocations be a function of tax rates (via individual optimality, market clearing and government budget balance). Then maximize utility with respect to the tax rates. The lack of uniqueness of the solution makes this impractical, but it is conceptually straightforward.

Our path will be more roundabout but nevertheless better. What we will do is to characterize the set of allocations that can be supported as a competitive equilibrium—for which there exist prices that, together with the given allocation, constitutes a competitive equilibrium.

Every allocation is not a competitive equilibrium allocation in this sense, even if it is resource feasible. To see this, consider what must be true of every competitive equilibrium. Denote by $p_t$ the after-tax price of good $t$ in terms of good 0 and by $w_t$ the after-tax relative price of leisure relative to consumption. We normalize $p_0 = 1.$

$$\sum_{t=0}^{\infty} p_t(c_t - w_t h_t) = a_0$$

where $a_0$ is the after-tax value of initial assets. We will see later that the government would like to tax these initial assets, but that amounts to a lump-sum tax which is (at least pragmatically) inconsistent with the idea that taxes have to be proportional to income or consumption (as opposed to assets). For this reason, it is tempting to regard $a_0$ as a parameter; otherwise, the optimal taxation problem might easily become trivial. Below we will see that it would be awkward to take this idea too literally and $a_0$ will in fact be subject to some manipulation.
Incidentally, a nice thing about the present value formulation is that we don’t have to bother with an explicit treatment of government debt. All we need to know at this stage is that it’s allowed. If we were really interested in it, we could back it out from the allocation and the first order conditions. (A perfect homework exercise!)

With this after-tax budget constraint, the first order conditions are simplified quite a bit. They are

\[ w_t u_{c,t} = u_{\ell,t} \]

and

\[ \frac{p_{t+1}}{p_t} = \beta \frac{u_{c,t+1}}{u_{c,t}}. \]

The second equation means that we can write

\[ p_t = \beta^t \frac{u_{c,t}}{u_{c,0}}. \]

The first (obviously) implies

\[ w_t = \frac{u_{\ell,t}}{u_{c,t}}. \]

Thus the consumer’s budget constraint becomes

\[ \sum_{t=0}^{\infty} \beta^t (u_{c,t} c_t - u_{\ell,t} h_t) = u_{c,0} a_0. \]  \hspace{1cm} (1)

Evidently this equation must be satisfied if an allocation is to be a part of any competitive equilibrium. We call it the “implementability” condition. The classic statement of this condition is in Lucas and Stokey (1983).

It turns out that something close to the converse is true as well: if an allocation satisfies (1) and the resource constraint, then the allocation is part of some competitive equilibrium. To show this, all we need to do is to come up with after-tax prices that are consistent with the given allocation being an optimal choice and the consumer’s intertemporal budget constraint is satisfied.

What about the government’s budget constraint? It is implied by the resource constraint and the consumer’s budget constraint, taken together. There is no need to impose all three. Now, then, is the time to introduce the resource constraint...s. There is one for every period, and this is significant.

\[ c_t + g_t + k_{t+1} = f(k_t, h_t) \]
where the sequence \((g_t)\) is exogenously given.

Now suppose we have an allocation, i.e. a sequence \((c_t)\) and a sequence \((h_t)\). What about \(k_t\)? We can back that out from the resource constraint, given \(k_0\). What is \(k_0\)? Apparently we have to take not only \(a_0\) as given but also \(k_0\). The most straightforward way of making these assumptions is to take initial government debt \(b_0\), initial capital \(k_0\) and the initial capital income tax \(\tau^k_0\) as given. Initial asset holdings then follow via

\[
a_0 = (1 - \tau^k_0)r_0(b_0 + k_0).
\]

Notice that the value of this expression can in fact be manipulated by the government through \(h_0\) which affects \(r_0\).

We now interrupt our discussion for a conceptual question. What did I mean when I said that an allocation must satisfy the resource constraint? Since we back out \((k_t)\) from the resource constraint, it would appear that every allocation satisfies the resource constraint for some appropriately chosen sequence \((k_t)\). Oh no it doesn’t! This is because we must have \(k_t \geq 0\) for all \(t\). If an allocation \((c_t), (h_t)\) implies \(k_{t+1} < 0\) for some \(t\) then we say that it is not resource feasible.

Now let’s construct after-tax prices so that our resource feasible allocation is part of a competitive equilibrium. This is almost ridiculously easy.

\[
w_t = \frac{u_{\ell,t}}{u_{c,t}}
\]

and

\[
p_t = \beta^t \frac{u_{c,t}}{u_{c,0}}.
\]

### 1.3 The Chamley-Judd result

Independently of one another, Chamley (1986) and Judd (1985) showed that intertemporal choices should not be distorted in the long run. That result may be described by saying that capital should not be taxed. But given tax equivalence, zero capital taxes involve no loss of generality so to say that capital should not be taxed is a vacuous statement. Meanwhile, the Chamley-Judd result is not vacuous. What it says is that, in the limit as \(t \to \infty\) we should have

\[
u_{c,t} = \beta u_{c,t+1} f_{k,t+1}.
\]
To see this, consider the Ramsey optimal taxation problem. It is to maximize
\[ \sum_{t=0}^{\infty} \beta^t u(c_t, \ell_t) \]
subject to
\[ h_t = 1 - \ell_t, \]
\[ c_t + g_t + k_{t+1} = f(k_t, h_t), \]
\[ k_{t+1} \geq 0, \]
\[ \sum_{t=0}^{\infty} \beta^t [u_{c,t} c_t - u_{\ell,t} h_t] = u_{c,0}(1 - \tau_k^0) r_0 (b_0 + k_0) \]
and \( k_0 \) and \( a_0 \) given.

Notice that although initial assets \( a_0 \) and \( k_0 \) are fixed, their “real value” in terms of current goods can be reduced by reducing \( r_0 = f_k(k_0, h_0) \), i.e. by encouraging an initial day of rest. Moreover, their value in terms of future goods can be reduced by reducing \( u_{c,0} \), i.e. encouraging an initial binge. More about that later.

Now denote the Lagrange multiplier associated with the “implementability” constraint by \( \lambda \) and define
\[ W(c, h; \lambda) = u(c, 1 - h) + \lambda [-u_c \cdot c + u_\ell \cdot h]. \]

Now form the Lagrangian:
\[ \mathcal{L} = \sum_{t=0}^{\infty} \beta^t W(c_t, h_t) + \sum_{t=0}^{\infty} \beta^t \mu_t [f(k_t, h_t) - c_t - k_{t+1}] + \lambda u_{c,0}(1 - \tau_k^0) r_0 (b_0 + k_0) \]

The first order conditions with respect to \( c_t \) are, for all \( t \neq 0 \),
\[ W_{c,t} = \mu_t \]
and with respect to \( k_{t+1} \) they are
\[ \mu_t = \beta f_{k,t+1} \mu_{t+1}. \]

Combining these familiar-looking results, we have the following equally familiar-looking result.
\[ W_{c,t} = \beta f_{k,t+1} W_{c,t+1}. \]
Now suppose consumption, leisure and the capital stock all converge to constants in the long run. Then $W_{c,t} = W_{c,t+1}$ and

$$1 = \beta f_k.$$  

Evidently this is exactly the same optimality condition as in the Pareto optimal allocation.

Under some conditions, intertemporal choices are not distorted as of period 1. That is true (why?) if

$$\frac{W_{c,t}}{u_{c,t}} = \frac{W_{c,t+1}}{u_{c,t+1}}.$$  

Under what conditions might this be true? Suppose

$$u(c_t, \ell_t) = c_t^{\frac{1-\sigma}{\sigma}} + v(\ell_t). \quad (2)$$

Then

$$W_{c,t} = [1 - \lambda(1 - \sigma)] c_t^{-\sigma}$$

and the condition is satisfied.

### 1.4 Backing out taxes and debt from the allocation

Given an allocation, it is often interesting to back out the policy that implements it. This cannot be done uniquely. To make the policy unique, it is necessary to impose restrictions. The “meta-theorem” is that you need precisely as many policy instruments as there are choice variables.

In the growth model there are three choice variables: consumption, leisure and investment. So let there be three instruments: a labor tax, a capital tax and government debt. (It matters that we allow for government debt. On the other hand, as we have seen, it doesn’t matter whether we allow for capital taxes or consumption taxes.)

Consider the consumer’s first order condition for labor supply, evaluated in a competitive equilibrium where pre-tax wages are equal to the marginal products of labor.

$$u_{\ell, t} = (1 - \tau^h_t) f_{h,t} u_{c,t}.$$  

This means that

$$\tau^h_t = 1 - \frac{u_{\ell, t}}{u_{c,t} f_{h,t}}.$$  

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Meanwhile,

\[ u_{c,t} = \beta (1 - \tau_{t+1}^k) f_{k,t+1} u_{c,t+1} \]

so that, for \( t > 0 \),

\[ \tau_t^k = 1 - \frac{u_{c,t-1}}{\beta f_{k,t} u_{c,t}}. \]

Finally, we can back out debt via

\[ b_{t+1} = \frac{1}{(1 - \tau_t^k) f_{k,t}} b_t + g_t - \tau_t^h f_{h,t} h_t - \tau_t^k f_{k,t} k_t. \]

### 1.5 Computation

What is the correct value of \( \lambda \)? Set \( \lambda = 0 \) and solve for the optimal allocation. Plug it into Equation (1) and see if it is satisfied. (If \( g_t \equiv 0 \) it will be, otherwise there is something wrong.) If not, increase \( \lambda \) a little bit and solve again, using your previous solution as an excellent initial guess. Keep going until Equation (1) is satisfied.

How to solve the model for a given \( \lambda \)? Here’s the best way to do that when the model is deterministic. What we do is treat all the equilibrium conditions as a system of non-linear equations and solve this system. Of course, there are infinitely many such equations. To deal with this problem, we assume that the steady state is reached after \( T \) periods, which is an excellent approximation if \( T \) is big enough.

The approach starts by computing the steady state. This is just a matter of solving a rather low-dimensional system of non-linear equations.

After that, solve for the transition. When computing it, it is useful to distinguish between “static” and “dynamic” equilibrium conditions. To define what I mean by that, denote the “state” by \( x_t \) and the other variables by \( d_t \). The “static” equilibrium conditions don’t involve \( d_{t+1} \), and let’s say that there are \( n_d \) of them. The “dynamic” conditions do involve \( d_{t+1} \) and there are \( n_x \) of them.

More precisely, let the equilibrium conditions be given by

\[
\begin{align*}
  f(x_t, x_{t+1}, d_t, d_{t+1}) &= 0 \\
  g(x_t, x_{t+1}, d_t) &= 0
\end{align*}
\]
With this notation in place, the equilibrium conditions are

\[
\begin{align*}
  g(x_0, x_1, d_0) &= 0 \\
  f(x_0, x_1, d_0, d_1) &= 0 \\
  g(x_1, x_2, d_1) &= 0 \\
  f(x_1, x_2, d_1, d_2) &= 0 \\
  \vdots \\
  f(x_{T-1}, x_T, d_{T-1}, d_T) &= 0 \\
  g(x_T, x_{T+1}, d_T) &= 0 \\
\end{align*}
\]

where \( x_0 \) is fixed by an initial condition and \( x_{T+1} \) is fixed by the steady state. What remains to solve for is \( x_1, x_2, \ldots, x_T \) and \( d_0, d_1, \ldots, d_T \). Notice that there are just as many equations as there are unknowns!

### 1.6 Balancing the budget

What if government debt is not allowed. This is a binding constraint. How do we impose it? After all, debt does not feature at all in the primal approach, so how on earth do we set it to zero? By translating into terms involving allocations only.

Apparently the budget is balanced if

\[
g_t = \tau^h_t f_{h,t} h_t + \tau^k_t f_{k,t} k_t
\]

Translating, we get

\[
g_t = [1 - \frac{u_{c,t}}{u_{c,t} f_{h,t}}] f_{h,t} h_t + [1 - \frac{u_{c,t-1}}{\beta f_{k,t} u_{c,t}}] f_{k,t} k_t
\]

Assuming constant returns to scale so that \( y_t = f(k_t, h_t) = f_{k,t} k_t + f_{h,t} h_t \), we have

\[
u_{c,t}(y_t - g_t) = u_{c,t} h_t + \beta^{-1} u_{c,t-1} k_t.
\]

Equivalently,

\[
u_{c,t} k_t = \beta [u_{c,t+1} (c_{t+1} + k_{t+1}) - u_{c,t} h_t].
\]

This can now be imposed as a constraint. It’s a non-standard constraint from the point of view of dynamic optimization theory. Specifically, it cannot be written as \( x_{t+1} = g(x_t, d_t) \). This raises some technical issues if you are interested in finding a feedback solution, which is the only feasible option if there is uncertainty. These technical issues are addressed, at a rather high level of generality, in Marcet and Marimon (1995).
2 Optimal taxation under uncertainty

2.1 With state-contingent debt

Consider an environment without capital where a representative consumer maximizes

$$E \left[ \sum_{t=0}^{\infty} \beta^t u(c_t, \ell_t) \right]$$

and where the resource constraint is

$$c_t + g_t = h_t = 1 - \ell_t$$

where \( \{g_t\} \) is a stochastic process living on the probability space \((\Omega, \mathcal{F}, P)\). Let \( \{\mathcal{F}_t\} \) be the filtration generated by \( \{g_t\} \). Markets are complete so that all contingent claims are associated with a well-defined price. Moreover, there are no limits on the quantities of contingent claims held in any particular period. Let \( \{Q_t : \mathcal{F} \to \mathbb{R}_+\} \) be a sequence of price measures with the following interpretation. The contingent claim that delivers 1 unit of consumption in period \( t \) if \( \omega \in A \) has price \( Q_t(A) \). We normalize so that \( Q_0(\Omega) = 1 \). Let \( W_t \) be the (after-tax) leisure price measure.

Meanwhile, define the stochastic processes \( p_t \) and \( w_t \) via

$$dQ_t = p_t dP \text{ on } \mathcal{F}_t$$

and

$$dW_t = w_t dP \text{ on } \mathcal{F}_t.$$ 

With these definitions, the consumer’s budget constraint (conceived as an equality) can be written as

$$E \left[ \sum_{t=0}^{\infty} [p_t c_t - w_t h_t] \right] = b_0.$$

By the consumer’s optimality conditions,

$$p_t = \beta^t \frac{u_{c,t}}{u_{c,0}}$$

and

$$w_t = -\beta^t \frac{u_{h,t}}{u_{c,0}}.$$
Hence the implementability condition becomes

\[
\mathbb{E} \left[ \sum_{t=0}^{\infty} \beta^t [u_{c,t} c_t - u_{\ell,t} h_t] \right] = u_{c,0} b_0.
\]

The equivalent sequence-of-markets representation of this condition is to assert the existence of an \( \{F_t\} \)-adapted stochastic process \( \{b_t\} \) with initial value \( b_0 \) and which satisfies

\[
\beta \mathbb{E}[u_{c,t+1} b_{t+1} | F_t] + u_{c,t} c_t - u_{\ell,t} h_t = u_{c,t} b_t
\]

for all \( t = 0, 1, 2, \ldots \) and

\[
\lim_{t \to \infty} \beta^t \mathbb{E} [u_{c,t} b_t] = 0.
\]

### 2.2 Without debt

The balanced budget constraint in this case becomes:

\[
u_{c,t} k_t = \beta \mathbb{E} [u_{c,t+1} (c_{t+1} + k_{t+1}) - u_{\ell,t+1} h_{t+1} | F_t].
\]

Can you derive it?

### 2.3 Without state-contingent debt

Aiyagari et al. (2002), in a paper based on Marcet et al. (1996), consider a world where debt is not state-contingent, i.e. a government bond must be a riskless asset whose return is determined one period in advance. This topic is also covered in Ljungqvist and Sargent (2000).

It is natural to impose these constraints on the sequence-of-markets representation rather than the present-value representation. We had

\[
\beta \mathbb{E}[u_{c,t+1} b_{t+1} | F_t] + u_{c,t} c_t - u_{\ell,t} h_t = u_{c,t} b_t
\]

and we now impose

\[
\mathbb{E}[b_{t+1} | F_t] = b_{t+1}.
\]

We get

\[
\beta \mathbb{E}[u_{c,t+1} | F_t] b_{t+1} + u_{c,t} c_t - u_{\ell,t} h_t = u_{c,t} b_t.
\]
In order for this predictability of debt to have any bite, we also impose
\[ b \leq b_{t+1} \leq \bar{b} \]
for each \( t = 0, 1, 2, \ldots \) though we will ignore this constraint, hoping that it doesn’t bind in equilibrium.

Incidentally, these bounds guarantee
\[ \lim_{t \to \infty} \beta^t \mathbb{E}[u_{c,t}b_t] = 0. \]

A particularly simple case is
\[ u(c, \ell) = c - \frac{1}{2}(1 - \ell_t)^2 \]

With these preferences, the government’s budget constraint is
\[ \beta b_{t+1} + c_t - h_t^2 = b_t. \]

Imposing the resource constraint \( c_t = h_t - g_t \), we have
\[ \beta b_{t+1} + h_t(1 - h_t) = b_t + g_t \]

Using the consumer’s foc, we get \( \tau_t^h = 1 - h_t \) and hence
\[ \beta b_{t+1} + \tau_t^h h_t = b_t + g_t \]
a result we might have figured out without so much fuss.

Now define the Hamiltonian via
\[ \mathcal{H} = \beta^t[h_t - g_t - \frac{1}{2}h_t^2] + \beta^{-1}\lambda_{t+1}[b_t - h_t + g_t + h_t^2]. \]

The optimality conditions are
\[ \beta^t[1 - h_t] + \beta^{-1}\mathbb{E}_{t}[\lambda_{t+1}][-1 + 2h_t] \]
and
\[ \mathbb{E}_{t}[\lambda_{t+1}] = \beta \lambda_t. \]

Now define
\[ \mu_t = \beta^{-t}\lambda_t. \]
We get

\[ 1 - h_t + E_t[\mu_{t+1}](-1 + 2h_t) = 0 \]

and

\[ E_t[\mu_{t+1}] = \mu_t \]

which means that \( \mu_t \) is a martingale. Meanwhile, we have

\[ 1 - h_t + \mu_t(2h_t - 1) = 0. \]

Solving for \( h_t \), we get

\[ h_t = \frac{\mu_t - 1}{2\mu_t - 1}. \]

What about the labour income tax rate? Apparently

\[ \tau_t^h = 1 - h_t \]

and so

\[ \tau_t^h = \frac{\mu_t}{2\mu_t - 1} \]

so that although the tax rate is not a martingale, it is a function of a martingale and therefore quite persistent, certainly more persistent than the underlying shock.
References


